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Introduction to the application of dynamical systems theory in the study of the dynamics of cosmological models of dark energy

Ricardo García-Salcedo¹, Tame Gonzalez²,
Francisco A Horta-Rangel², Israel Quiros^{2,3} and
Daniel Sanchez-Guzmán¹

¹CICATA— Legaria del Instituto Politécnico Nacional, 11500, México, D.F., Mexico

²Departamento de Ingeniería Civil, División de Ingeniería, Universidad de Guanajuato, Gto., Mexico

E-mail: rigarcias@ipn.mx, tamegc72@gmail.com, anthort@hotmail.com,
iquiros6403@gmail.com and dsanchezgz@gmail.com

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Abstract

The theory of dynamical systems is a very complex subject that has produced several surprises in the recent past in connection with the theory of chaos and fractals. The application of the tools of dynamical systems in cosmological settings is less known, in spite of the number of published scientific papers on this subject. In this paper, a mostly pedagogical introduction to the cosmological application of the basic tools of dynamical systems theory is presented. It is shown that, in spite of their amazing simplicity, these tools allow us to extract essential information on the asymptotic dynamics of a wide variety of cosmological models. The power of these tools is illustrated within the context of the so-called Λ -cold dark matter (ACDM) and scalar field models of dark energy. This paper is suitable for teachers, undergraduate students, and post-graduate students in the disciplines of physics and mathematics.

Keywords: science teaching, cosmology, nonlinear dynamics and chaos

1. Introduction

Cosmology is the study of the structure and evolution of the universe as a whole. Einstein's general relativity (GR) stands as its mathematical basis. In general, Einstein's GR equations are a very complex system of coupled, nonlinear differential equations, but the system is

³ Author to whom any correspondence should be addressed.

simplified by underlying spacetime symmetries. The Friedmann–Robertson–Walker (FRW) spacetime model is depicted by the line element⁴

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - k r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where t is the cosmic time, r , θ , and ϕ are comoving spherical coordinates, $a(t)$ is the cosmological scale factor, and the constant, $k = -1, 0, +1$, parametrizes the curvature of the spatial sections. This model is based on what is known as the cosmological principle: at large scales, the Universe is homogeneous and isotropic. Homogeneity and isotropy imply that there is no preferred place and no preferred direction in the universe⁵. These observations confirm the validity of the cosmological principle at large scales ($\sim 100 \text{ Mpc} \approx 10^{26} \text{ cm}$). The dynamics of the universe is fully given by the explicit form of the scale factor, which depends on the symmetry properties and the matter content of the Universe.

To determine the scale factor, it is necessary to solve Einstein's cosmological field equations, which relate the spacetime geometry with the distribution of matter in the Universe. The maximal symmetry implied by the cosmological principle allows for great simplification of Einstein's equations. In this paper, for further simplification, we shall consider FRW spacetimes with flat spatial sections ($k = 0$). The resulting cosmological equations are written as follows⁶:

$$\begin{aligned} 3H^2 = \sum_x \rho_x + \rho_X, \quad \dot{H} = -\frac{1}{2} \left(\sum_x \gamma_x \rho_x + \gamma_X \rho_X \right), \\ \dot{\rho}_x + 3\gamma_x H \rho_x = 0, \quad \dot{\rho}_X + 3\gamma_X H \rho_X = 0, \end{aligned} \quad (1)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, ρ_x and ρ_X are the energy densities of the gravitating matter and of the X -fluid (the dark energy in this paper), respectively, and the sum is over all of the gravitating matter species living in the FRW spacetime (dark matter, baryons, radiation, etc.). The last two equations above are the conservation equations for the gravitating matter degrees of freedom and for the X -fluid, respectively⁷. In addition, it is adopted that the following equations of state are obeyed:

$$p_x = (\gamma_x - 1)\rho_x, \quad p_X = (\gamma_X - 1)\rho_X,$$

where p_x is the barotropic pressure of the x -matter fluid and p_X is the parametric pressure of the X -component⁸, while γ_x and γ_X are their barotropic parameters⁹.

According to our current understanding of expansion history, the Universe was born out of a set of initial conditions known as the big bang. Further evolution led to a stage of primeval inflation driven by a scalar field [4]. The inflation era was followed by a stage dominated by radiation, which was followed in turn, by an intermediate stage dominated by

⁴ For a pedagogical exposition of the geometrical and physical significance of the FRW metric, see [1].

⁵ Here we adopt the units where $8\pi G = c = 1$.

⁶ For a didactic derivation of the cosmological equations not based on GR, see [1–3].

⁷ It should be pointed out that one equation in (1), say the second (Raychaudhuri) equation, is redundant.

⁸ By parametric pressure, we understand that it might not correspond to standard barotropic pressure with the usual thermodynamic properties, but that it is rather a conveniently defined parameter that obeys similar equations to its matter counterpart.

⁹ In several places in this paper, we use the so-called equation of state (EOS) parameter, $\omega_X = p_X/\rho_X = \gamma_X - 1$, interchangeably with the barotropic parameter, γ_X .

nonrelativistic cold dark matter (CDM)¹⁰. During this stage of the cosmic expansion, most of the structure we observe was formed. The end of the matter-domination era can be traced back to a recent moment of the cosmic expansion when a very peculiar form of matter known as dark energy (DE), which does not interact with baryons, radiation, or any other form of ‘visible’ matter, started dominating [5]. This alien form of matter antigravitates and causes the present Universe to expand at an accelerating pace instead of decelerating, as it would for a matter-dominated (either CDM or baryons) or radiation-dominated Universe. Actually, if one substitutes

$$\frac{\ddot{a}}{a} = \dot{H} + H^2$$

back into the second equation in (1) and takes into account the Friedmann equation (the first equation in (1)) one is left with the following equation:

$$\frac{\ddot{a}}{a} = -\frac{1}{2} \sum_x \left(\gamma_x - \frac{2}{3} \right) \rho_x - \frac{1}{2} \left(\gamma_X - \frac{2}{3} \right) \rho_X. \quad (2)$$

It is known that for standard (gravitating) forms of matter, the barotropic pressure is a non-negative quantity: $p_x = (\gamma_x - 1)\rho_x \geq 0 \Rightarrow \gamma_x \geq 1$. In particular, for CDM and baryons, the (constant) barotropic index $\gamma_m = 1$, while for radiation $\gamma_r = 4/3$. Hence, since obviously $\gamma_x > 2/3$ —if we forget for awhile about the second term on the right-hand side (rhs) of (2)—one can see that a cosmic background composed of standard matter will expand at a decelerated pace ($\ddot{a} < 0$). In contrast, for unconventional matter with $\gamma_X < 2/3$ (the parametric pressure p_X is obviously negative), the second term on the rhs of equation (2) contributes to accelerating the pace of the cosmological expansion. In particular, if we choose the X -component in the form of vacuum energy, $(\gamma_{\text{vac}}, \rho_{\text{vac}}) \rightarrow (\gamma_X, \rho_X)$, usually identified with the energy density of the cosmological constant (see section 3), since $p_{\text{vac}} = -\rho_{\text{vac}}$, then $\gamma_{\text{vac}} = 0 < 2/3$. Due to the known fact that the vacuum energy is not diluted by the cosmic expansion¹¹, eventually this component of the cosmic budget will dominate the cosmological dynamics, leading to late-time acceleration of the expansion

$$\frac{\ddot{a}}{a} \approx \frac{1}{3} \rho_{\text{vac}} > 0.$$

A related (dimensionless) parameter that measures whether the expansion is accelerated or decelerated is the so-called deceleration parameter, which is defined as

$$q \equiv -1 - \frac{\dot{H}}{H^2} = -\frac{a\ddot{a}}{\dot{a}^2}. \quad (3)$$

The pace of the expansion accelerates if $q < 0$. On the contrary, positive $q > 0$ means that the expansion is decelerating. This parameter will be useful in the discussion in the following sections.

In spite of the apparent simplicity of equation (1), as with any system of nonlinear second-order differential equations, it is a very difficult (and perhaps unsuccessful) task to find exact solutions. Even when an analytic solution can be found, it will not be unique, but rather just one in a large set of solutions [6]. This is in addition to the question about the

¹⁰ Dark matter is a conventional form of matter in the sense that it gravitates just like any other known sort of standard matter such as radiation, baryons, etc. However, unlike the latter forms of matter, CDM does not interact with radiation, which is why it is called ‘dark matter.’

¹¹ We recall that, in contrast to the vacuum energy density, which does not dilute with the expansion $\rho_{\text{vac}} = \Lambda = \text{const}$, the energy density of CDM and baryons dilutes like $\propto a^{-3}$, while the density of radiation decays very quickly $\propto a^{-4}$.

stability of given solutions. In this case, the dynamical systems tools come to our rescue. These very simple tools allow us to correlate important concepts like past and future attractors (and saddle equilibrium points) in the phase space, with generic solutions to the set of equation (1) without the need to analytically solve them.

In correspondence with the previously mentioned stages of the cosmic expansion, one expects that the state space and the phase space of any feasible cosmological model should be characterized by at least one of the following equilibrium configurations in the space of cosmological states: (i) a scalar-field-dominated (inflationary) past attractor, (ii) saddle critical points associated with radiation-domination, radiation-matter scaling, and matter-domination, and either (iii) a matter/DE scaling or (iv) a de Sitter¹² future attractor. Any heteroclinic orbit joining these critical points then represents a feasible image of the featured cosmic evolution in the phase space¹³.

While the theory of dynamical systems has produced several surprises in the recent past in connection with the theory of chaos [7, 8] and fractals [9], the application of the tools of dynamical systems in cosmological settings is less known, in spite of the number of published scientific papers on this subject (to cite few, see [2, 5, 10–19] and the related references therein).

This paper presents an introduction to the application of the simplest tools of the theory of dynamical systems—those that can be explained with only previous knowledge of the fundamentals of linear algebra and of ordinary differential equations—within the context of the so-called ‘concordance’ or Λ CDM (section 3) and scalar field models (section 4) of dark energy [5, 11–16, 18–22]. The dynamical systems tools play a central role in the understanding of the asymptotic structure of these models. These tools are helpful in looking for an answer to questions like where does our Universe come from? What would be its fate? As a specific example, the cosh potential is explored in section 5, to show the power of these tools in the search for generic dynamical behaviour.

We want to stress that there are very good introductory and review papers on the application of dynamical systems in cosmology—see, for instance, [5, 10–12, 16]—so, what is the point of writing another introductory paper on the issue? We want to point out that in the present paper, we aim not only to provide another introductory exposition of the essentials of the subject, but also to fill several gaps regarding specific topics not usually covered in similar publications. Here we pay special attention to the necessary (yet usually forgotten) definition of the physically meaningful phase space (i.e., the region of the phase space that contains physically meaningful cosmological solutions). We also include the derivation of relevant formulas to pave the way for beginners in their progress towards concrete computations. We shall explain, in particular, a method developed in [23] to consider arbitrary potentials within the dynamical systems study of scalar field cosmological models. This subject is not usually covered in the existing introductory literature. For completeness, at the end of the paper we include an appendix (section appendix) where an elementary introduction to the theory of the dynamical systems is given and useful comments on the interplay between the cosmological field equations and the equivalent phase space are also provided. Only knowledge of the fundamentals of linear algebra and of ordinary differential equations is required to understand the material in the appendix. The exposition in the appendix is

¹² A de Sitter cosmological phase is a stage of the cosmological expansion whose dynamics is described by the line element $ds^2 = -dt^2 + e^{H_0 t} dx^2$, where H_0 is the Hubble constant. Since the deceleration parameter, $q \equiv -1 - \dot{H}/H^2 = -1$, is a negative quantity, the de Sitter expansion is inflationary.

¹³ For very concise introductory information on the theory of dynamical systems and related concepts like critical points (attractors and saddle points), heteroclinic orbits, etc, see the appendix.

complete enough to allow one to understand how the computations in sections 3 and 4 are done. We recommend that readers who are not familiar with dynamical systems theory start with this section.

Our goal is to keep the discussion as general as possible while presenting the exposition in as pedagogical a way as possible. The paper contains many footnotes with comments and definitions that complement the main text. Since the subject is covered with some degree of technical detail, and since a basic knowledge of cosmology is assumed, this paper is suitable for teachers, undergraduate students, and postgraduate students, from the disciplines of physics and mathematics.

2. Asymptotic cosmological dynamics: a model-independent analysis

In this section, the asymptotic structure in the phase space of a general DE model will be discussed in detail. For simplicity and compactness of the exposition, we shall assume only a two-component cosmological fluid composed of CDM, which is labelled here by the index ‘ m ,’ (i.e., $(\rho_m, p_m) \rightarrow (\rho_x, p_x)$) and of DE (the X -component).

To derive an autonomous system of ordinary differential equations (ASODE) out of (1), it is useful to introduce the so-called dimensionless energy density parameters of matter and of the DE

$$\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_X = \frac{\rho_X}{3H^2}, \quad (4)$$

respectively, which are always non-negative quantities¹⁴. In terms of these parameters, the Friedmann equation—the first equation in (1)—can be written as the following constraint:

$$\Omega_m + \Omega_X = 1, \quad (5)$$

which entails that none of the non-negative dimensionless energy density components alone may exceed unity: $0 \leq \Omega_m \leq 1$, $0 \leq \Omega_X \leq 1$.

Given the above constraints, if one thinks of the dimensionless density parameters as variables of some state space, only one of them is linearly independent, say Ω_X . Let us write a dynamical equation for Ω_X , where we rewrite the Raychaudhuri equation—the second equation in (1)—and the conservation equation for the X -component, in terms of Ω_X :

$$\begin{aligned} 2\frac{\dot{H}}{H^2} &= -3\gamma_m\Omega_m - 3\gamma_X\Omega_X \\ &= -3\gamma_m + 3(\gamma_m - \gamma_X)\Omega_X, \\ \frac{\dot{\rho}_X}{3H^2} &= -3\gamma_X H\Omega_X. \end{aligned} \quad (6)$$

Then, if we substitute (6) into

$$\dot{\Omega}_X = \frac{\dot{\rho}_X}{3H^2} - 2\frac{\dot{H}}{H}\Omega_X,$$

and consider constraint (5), we obtain

$$\Omega'_X = 3(\gamma_m - \gamma_X)\Omega_X(1 - \Omega_X), \quad (7)$$

¹⁴ In agreement with the conventional non-negativity of energy, we consider non-negative energy densities exclusively.

where the tilde denotes the derivative with respect to the new variable, $\tau \equiv \ln a$ ($\dot{\Omega} = H\Omega'$, etc).

In general, for varying γ_X , another ordinary differential equation (ODE) for γ_X is needed to have a closed system of differential equations (the barotropic index of matter, γ_m , is usually set to a constant). The problem is that, unlike the ODE (7), which is model-independent, the ODE for $\gamma_X = (\rho_X + p_X)/\rho_X$ requires certain specifications that depend on the chosen model (see below).

Certain important results may be extracted from (7) under certain assumptions. For instance, if we assume a constant γ_X , the previously mentioned ODE is enough to uncover the asymptotic structure of the model (1) in the phase space. In this case, the phase space is the one-dimensional segment, $\Omega_X \in [0, 1]$. The critical points are

$$\Omega_X(1 - \Omega_X) = 0 \Rightarrow \Omega_X = 1, \Omega_X = 0,$$

where we assume that $\gamma_X \neq \gamma_m$. The first fixed point, $\Omega_X = 1$, is correlated with DE domination, while the second one, $\Omega_X = 0$ ($\Omega_m = 1$), is associated with CDM-dominated expansion.

If we linearize equation (7) around the critical points, $1 - \delta_1 \rightarrow 1$ and $0 + \delta_0 \rightarrow 0$ (δ_1 and δ_0 are small perturbations), one obtains

$$\begin{aligned} \delta_1' &= -3(\gamma_m - \gamma_X)\delta_1 \Rightarrow \delta_1(\tau) = \bar{\delta}_1 e^{-3(\gamma_m - \gamma_X)\tau}, \\ \delta_0' &= 3(\gamma_m - \gamma_X)\delta_0 \Rightarrow \delta_0(\tau) = \bar{\delta}_0 e^{3(\gamma_m - \gamma_X)\tau}. \end{aligned} \quad (8)$$

As seen in the DE-dominated solution,

$$\Omega_X = 1 \Rightarrow 3H^2 = \rho_X \propto a^{-3\gamma_X} \Rightarrow a(t) \propto t^{2/3\gamma_X}$$

is a stable (isolated) equilibrium point whenever $\gamma_m > \gamma_X$ since, in this case, the perturbation

$$\delta_1(\tau) \propto e^{-3(\gamma_m - \gamma_X)\tau},$$

decreases exponentially. Hence, given that $\gamma_m > \gamma_X$, the DE-dominated solution, $\Omega_X = 1$, is the future attractor solution that describes the fate of the Universe. Besides, if $\gamma_m > \gamma_X$, the matter-dominated solution, $\Omega_m = 1$ ($\Omega_X = 0$): $a(t) \propto t^{2/3\gamma_m}$, is the past attractor since the initially small perturbation, $\delta_0(\tau) \propto e^{3(\gamma_m - \gamma_X)\tau}$, increases exponentially with τ , thus taking the system away from the condition $\Omega_m = 1$. Otherwise, if $\gamma_m < \gamma_X$, the X -dominated solution is the past attractor while the matter-dominated solution is the future attractor. However, this last situation is not consistent with the known cosmic history of our Universe.

The interesting thing about the above result is that it is model-independent (i.e., no matter which model we adopt for the DE (X -component), given that γ_X is a constant, the above mentioned critical points and their stability properties are obtained). Besides, since a non-constant, γ_X , obeys an autonomous ODE of the general form $\gamma_X' = f(\gamma_X, \Omega_X)$, at any equilibrium point where, necessarily

$$\Omega_X' = 0, \gamma_X' = 0 \Rightarrow \gamma_X = \bar{\gamma}_X = \text{const}$$

provided that $\bar{\gamma}_X \neq \gamma_m$, either $\Omega_X = 1$, or $\Omega_X = 0$ ($\Omega_m = 1$). Hence, independent of the model adopted to account for the X -component, matter-dominated and X -fluid-dominated solutions are always equilibrium points in the phase space of the two-fluids cosmological model depicted by equations (1). This is a generic result.

3. Λ CDM model

With the help of a specific cosmological model of DE, we will now show how it is possible that a two-dimensional system of ODEs may come out of the three-dimensional system of second-order field equations (1) where, we recall, one of these equations is redundant. For this purpose, we will focus on the so-called Λ -cold dark matter (Λ CDM) model [24], whose phase space is a subspace of the phase plane¹⁵. Here, and only for the extent of this section, in addition to the CDM and to the DE (the cosmological constant), we will consider a radiation fluid with energy density ρ_r and pressure $p_r = \rho_r/3$ ($\gamma_r = 4/3$). The conservation equation for radiation reads

$$\dot{\rho}_r + 4H\rho_r = 0 \Rightarrow \rho_r \propto a^{-4}.$$

Meanwhile, for the cosmological constant term, as for any vacuum fluid, we have $p_\Lambda = -\rho_\Lambda$ ($\gamma_\Lambda = 0$), which means that $\dot{\rho}_\Lambda = 0$, so this term does not evolve during the course of the cosmic expansion.

In what follows, since we adopted the units system with $8\pi G = c = 1$, we write $\rho_\Lambda = \Lambda$. Besides, since we deal here with CDM, which is modelled by a pressureless dust, we set $\gamma_m = 1$. After these assumptions, and identifying $\rho_X \equiv \rho_\Lambda = \Lambda$ in equation (1), the resulting cosmological equations are

$$\begin{aligned} 3H^2 &= \rho_r + \rho_m + \Lambda, \quad \dot{H} = -\frac{1}{2}\left(\frac{4}{3}\rho_r + \rho_m\right), \\ \dot{\rho}_m + 3H\rho_m &= 0, \quad \dot{\rho}_r + 4H\rho_r = 0. \end{aligned} \quad (9)$$

At this point, let us explore how the cosmological constant can explain the present speed-up of the cosmic expansion. If we conveniently combine the first two equations above, taking into account that straightforward integration of the last equations in (9) yields $\rho_m = 6C_m/a^3$ and $\rho_r = 3C_r/a^4$, respectively (C_m and C_r are constants), we get

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{C_r}{a^4} - \frac{C_m}{a^3} + \frac{\Lambda}{3}.$$

We see that acceleration of the expansion $\ddot{a} > 0$ may occur in this model thanks to the third term above (the DE, i.e., the cosmological constant). Besides, the accelerated pace of the expansion is a recent phenomenon: during the course of the cosmic evolution, the initially dominating radiation component dilutes very quickly, $\propto a^{-4}$, until the matter component starts dominating. As the cosmic expansion further proceeds, the matter component also dilutes, $\propto a^{-3}$, until very recently, when the cosmological constant started to dominate to yield to positive $\ddot{a} > 0$.

Following the same procedure as in the preceding section, we write the Friedmann constraint

$$\Omega_r + \Omega_m + \Omega_\Lambda = 1, \quad (10)$$

where $\Omega_r = \rho_r/3H^2$ and $\Omega_\Lambda = \Lambda/3H^2$. In what follows, we choose the following variables of the phase space:

$$x \equiv \Omega_r, \quad y \equiv \Omega_\Lambda, \quad (11)$$

¹⁵ For a dynamical systems study of the FRW cosmological model with spatial curvature $k \neq 0$ and with a cosmological constant, see [2].

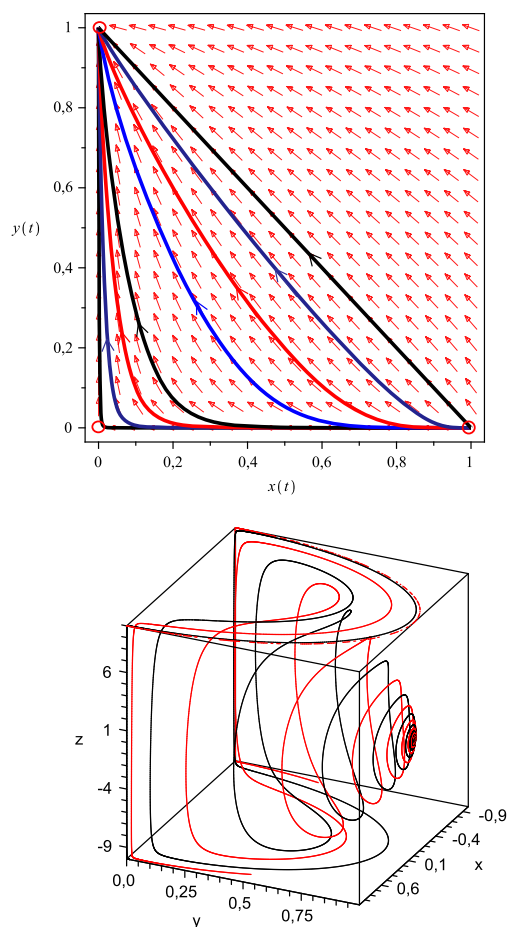


Figure 1. Phase portrait of the ASODE (13) corresponding to the Λ CDM model (top panel). The orbits of (13) emerge from the past attractor, $P_r: (1, 0)$ —the radiation domination state—and end up in the future attractor, $P_{dS}: (0, 1)$ (de Sitter phase). For a non-empty set of initial conditions, the corresponding orbits approach close enough to the matter-dominated saddle point at the origin, $P_m: (0, 0)$. The bottom panel shows the phase portrait of the ASODE (31), which presents two orbits generated by two different sets of initial conditions. Here, the free parameter, μ , has been arbitrarily set at $\mu = 10$.

where, for the sake of simplicity of writing, we adopt x and y in place of the dimensionless energy densities of radiation, Ω_r , and of the cosmological constant, Ω_Λ , respectively. But we warn the reader that the same symbols, x and y , will be used in the next sections to represent different variables of the phase space.

We have that $\Omega_m = 1 - x - y$, and, since $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq \Omega_m \leq 1$, the physically relevant phase space is defined as the following two-dimensional triangular region (see the top panel of figure 1):

$$\Psi_\Lambda = \{(x, y): x + y \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1\}. \quad (12)$$

As before, to derive the autonomous ODEs, we first take the derivative of the variables x and y with respect to the cosmic time, t :

$$\dot{x} = \frac{\dot{\rho}_r}{3H^2} - 2\frac{\rho_r}{3H^2} \frac{\dot{H}}{H}, \quad \dot{y} = -2\frac{\Lambda}{3H^2} \frac{\dot{H}}{H},$$

where we have to keep in mind the Raychaudhuri equation and the conservation equation for radiation in (9), written in terms of the variables of the phase space

$$\dot{H} = -\frac{H^2}{2}(3 + x - 3y),$$

and then we have to go to derivatives with respect to the variable $\tau = \ln a(t)$ ($\xi' \rightarrow H^{-1}\dot{\xi}$). We obtain

$$x' = -x(1 - x + 3y), \quad y' = (3 + x - 3y)y. \quad (13)$$

The simplicity of the system of two ordinary differential equation (13) is remarkable when compared with the system of three second-order cosmological equation (9). The critical points of (13) in \mathcal{P}_Λ are easily found by solving the following system of algebraic equations: $x(1 - x + 3y) = 0$, $(3 + x - 3y)y = 0$.

In the present case, in terms of the variables x , y , the deceleration parameter, $q = -1 - \dot{H}/H^2$ (equation (3)), can be written as

$$q = (1 + x - 3y)/2,$$

which means that those fixed points that fall in the region of the phase space lying above the line, $y = 1/3 + x/3$, correspond to cosmological solutions where the expansion is accelerating.

3.1. Critical points

3.1.1. Radiation-dominated critical point. The critical point, $P_r: (1, 0)$, $\Omega_r = 1 \Rightarrow 3H^2 = \rho_r \Rightarrow a(t) \propto \sqrt{t}$, corresponds to the radiation-dominated cosmic phase. This solution depicts decelerated expansion, since $q = 1$. The linearization matrix for the system of ODE (13) (for details, see the appendix) is

$$J = \begin{pmatrix} -1 + 2x - 3y & -3x \\ y & 3 + x - 6y \end{pmatrix}.$$

The roots of the algebraic equation

$$\det |J(P_r) - \lambda U| = \det \begin{pmatrix} 1 - \lambda & -3 \\ 0 & 4 - \lambda \end{pmatrix} = 0,$$

are the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Since both eigenvalues are positive reals, then P_r is a source point (past attractor). This means that in the Λ CDM model described by the cosmological equation (9), the radiation-dominated solution, $a(t) \propto \sqrt{t}$, is a privileged solution.

From the inspection of the phase portrait of (13), seen in the top panel of figure 1, it is apparent that any viable pattern of cosmological evolution should start in a state where the matter content of the Universe is dominated by radiation. Of course this is a drawback of our classical model, which cannot explain the very early stages of the cosmic evolution, where the quantum effects of gravity play a role. This includes the early inflation. In a model that would account for this early period of the cosmic dynamics, inflation should be the past attractor in the equivalent phase space.

3.1.2. Matter-dominated critical point. The critical point, $P_m: (0, 0)$, $\Omega_m = 1 \Rightarrow 3H^2 = \rho_m \Rightarrow a(t) \propto t^{2/3}$, corresponds to the matter-dominated solution, which is associated with decelerated expansion ($q = 1/2$). Following the same procedure above (see the appendix) one finds the following eigenvalues of the linearization matrix, $J(P_m)$, evaluated at the hyperbolic equilibrium point, P_m : $\lambda_1 = 1$, $\lambda_2 = -3$. Hence, this is a saddle critical point. As seen in the phase portrait in the top panel of figure 1, only with conveniently chosen initial conditions do the given orbits in Ψ_Λ approach close enough to P_m . Since this is an unstable (metastable) critical point, it can only be associated with a transient stage of the cosmic expansion. This is good, since a stage dominated by dark matter can only be a transient state, lasting for enough time to account for the observed amount of cosmic structure. A drawback is that we need to fine tune the initial conditions in order for feasible orbits in the phase space to get close enough to P_m .

3.1.3. de Sitter phase. The critical point, $P_{dS}: (0, 1)$, $\Omega_\Lambda = 1 \Rightarrow 3H^2 = \Lambda \Rightarrow a(t) \propto e^{\sqrt{\Lambda/3}t}$, corresponds to a stage of inflationary ($q = -1$) de Sitter expansion. The eigenvalues of the matrix $J(P_{dS})$ are $\lambda_1 = -3$ and $\lambda_2 = -4$. This means that P_{dS} is a future attractor (see the classification of isolated equilibrium points in the appendix). This entails that, independent of the initial conditions chosen, Ω_r^0 , Ω_Λ^0 , the orbits in Ψ_Λ are always attracted to the de Sitter state, which explains the actual accelerated pace of the cosmic expansion in perfect fit with the observational data. This is why, in spite of the serious drawbacks in connection with the cosmological constant problem [25, 26], the very simple Λ CDM model represents such a successful description of the present cosmological paradigm and therefore is called the ‘concordance model.’ Any other DE model, regardless of its nature, ought to be compared to the Λ CDM predictions as a first viability test.

Additional refinement of the above model is achieved if we add the energy density of baryons.

4. Scalar field models of dark energy: the dynamical systems perspective

The cosmological constant problem can be split into two questions [25]. (i) Why isn’t the vacuum energy $\rho_{vac} = \Lambda$ much larger? (ii) Why it is of the same order of magnitude as the present mass density of the Universe? The first question is an old cosmological constant problem, and the second is acknowledged as the new cosmological constant problem. To avoid the old cosmological constant problem, which is exclusive to the Λ CDM model, scalar field models of dark energy are invoked¹⁶. In this last case, an effective ‘dynamical’ cosmological constant is described by the scalar field’s X self-interaction potential, $V(X)$. This may be arranged so that at early times, the vacuum energy, $\rho_{vac,0} = V(X_0)$ ($\dot{X} = 0$), is large enough to produce the desired amount of inflation, while at late times, $\rho_{vac,f} = V(X_f)$ is of the same order of the CDM energy density, ρ_m .

In this section, we will explore the asymptotic structure of general scalar field models of DE [5, 11–16, 18–22, 27, 28]. These structures represent a viable alternative to the Λ CDM model explored above. Among them, ‘exponential quintessence,’ $V(X) = M \exp(-\mu X)$ [11, 29], is one of the most popular scalar field models of DE.

The specification of a scalar field model for the DE means that the energy density and the parametric pressure in the cosmological field equation (1) are given by

¹⁶ The new cosmological problem is also inherent in several scalar field models of dark energy.

$$\rho_X = \dot{X}^2/2 + V(X), \quad p_X = \dot{X}^2/2 - V(X), \quad (14)$$

respectively. In these equations, $V = V(X)$ is the self-interacting potential of the scalar field, X . In addition, for the equation of state parameter $\omega_X = \gamma_X - 1$, one has

$$\omega_X \equiv \frac{p_X}{\rho_X} = \frac{\dot{X}^2 - 2V}{\dot{X}^2 + 2V}. \quad (15)$$

After the above choice, the conservation equation (1) can be written in the form of the following Klein-Gordon equation:

$$\ddot{X} + 3H\dot{X} = -V_{,X}. \quad (16)$$

It happens that deriving an autonomous ODE for the variable ω_X (see section 2) can be a very difficult task. Hence, it could be better to choose a different set of phase space variables to do the job. Here, instead of the phase space variables Ω_X and ω_X , we choose the variables [11]

$$x \equiv \frac{\dot{X}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \quad (17)$$

so that the dimensionless energy density of the scalar field, X , is given by $\Omega_X = x^2 + y^2$, and the Friedmann constraint (10) can be written as

$$\Omega_m = 1 - x^2 - y^2. \quad (18)$$

4.1. Determination of the physical phase space

The first step toward a complete study of the asymptotic structure of a given cosmological model is the rigorous determination of the phase space where we look for relevant equilibrium points. In the present model, there are several constraints on the physical parameters that help us define the physically meaningful phase space.

One of these constraints is given by (18)

$$0 \leq \Omega_m \leq 1 \Rightarrow 0 \leq x^2 + y^2 \leq 1.$$

Also, since $0 \leq \Omega_X \leq 1$, then $|x| \leq 1$. Additionally, we will be interested in cosmic expansion exclusively, $H \geq 0$, so that $y \geq 0$. For instance, for potentials of one of the following kinds (see below), $V = V_0$, $V = V_0 e^{\pm\mu X}$, the physical phase plane is defined as the upper semi-disc,

$$\mathcal{P}_X := \{(x, y): 0 \leq x^2 + y^2 \leq 1, |x| \leq 1, y \geq 0\}. \quad (19)$$

However, as we will see, in general (arbitrary potentials) the phase space is of a dimension higher than 2 [20].

4.2. Autonomous system of ODEs

To derive the autonomous ordinary equations for the phase space variables x, y , one proceeds in a similar fashion as in section 2. First, we write the Raychaudhuri equation in (1)

$$\begin{aligned}\dot{H} &= -\frac{1}{2}(\gamma_m \rho_m + \rho_X + p_X) \\ \Rightarrow -2\frac{\dot{H}}{H^2} &= 3\gamma_m \Omega_m + \frac{\dot{X}^2}{H^2},\end{aligned}$$

in terms of the new variables:

$$-2\frac{\dot{H}}{H^2} = 3\gamma_m(1 - x^2 - y^2) + 6x^2, \quad (20)$$

where we have taken into account the Friedmann constraint (18). Then, given definition (17), let us find

$$\dot{x} = \frac{\ddot{X}}{\sqrt{6}H} - \frac{\dot{X}}{\sqrt{6}H} \frac{\dot{H}}{H}, \quad (21)$$

or if one substitutes (16) and (20) back into (21), then

$$x' = -3x(1 - x^2) + \frac{3\gamma_m}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\frac{V_{,X}}{V}y^2, \quad (22)$$

where we replaced derivatives with respect to the cosmic time, t , with derivatives with respect to the variable $\tau \equiv \ln a(t)$. Applying the same procedure with the variable y , one obtains

$$y' = y\left(\frac{1}{2}\frac{V_{,X}}{V}\frac{\dot{X}}{H} - \frac{\dot{H}}{H^2}\right),$$

where we have taken into account that $\dot{V} = V_{,X}\dot{X}$; hence

$$y' = y\left[\frac{3\gamma_m}{2}(1 - x^2 - y^2) + 3x^2\right] + \sqrt{\frac{3}{2}}\frac{V_{,X}}{V}xy. \quad (23)$$

Unless $V = V_0 \Rightarrow V_{,X}/V = 0$, or $V = V_0 e^{\pm\mu X} \Rightarrow V_{,X}/V = \pm\mu = \text{const}$, the ASODEs (22) and (23) are not a closed system of equations since one equation involving the derivative of $V_{,X}/V$ with respect to τ is lacking.

One example where the ASODEs (22) and (23) are indeed closed systems of ODEs is given by the so-called ‘exponential quintessence’. This case has been investigated in detail in [11], so we point the reader to that reference to look for a very interesting example of the application of the dynamical systems tools in cosmology. As a matter of fact, reproducing the results of the dynamical systems study in [11] is a very useful exercise for those who want to learn how to apply dynamical systems tools in cosmological settings.

To be able to consider self-interaction potentials beyond the constant and the exponential potentials, in addition to variables x , y , one needs to adopt a new variable [14, 23, 30]

$$z \equiv V_{,X}/V, \quad (24)$$

so that $z = 0$ corresponds to the constant potential, while $z = \pm\mu$ for the exponential potential¹⁷. By taking the derivative of the new variable, z , with respect to τ , one obtains

¹⁷ For non-exponential potentials, in addition to the increase in the dimensionality of the phase space, there is another problem. Typically the phase space becomes unbounded, so that two different sets of variables are required to cover the entire phase space [20]. In the present paper, however, the chosen example of the cosh-like potential does not lead to unbounded phase space. In consequence, a single set of phase space variables is enough to describe the whole asymptotic (and intermediate) dynamics.

$$z' = \sqrt{6}xf(z), \quad (25)$$

where we have defined (as before, take into account that $\dot{V} = V_X \dot{X}$ and $\dot{V}_X = V_{,XX} \dot{X}$, etc)

$$f(z) \equiv z^2[\Gamma - 1], \quad \Gamma \equiv VV_{,XX}/V_{,X}^2, \quad (26)$$

and the main assumption has been that the above Γ is a function of the variable z , $\Gamma = \Gamma(z)$. Note that, since

$$\Gamma \equiv \frac{VV_{,XX}}{V_{,X}^2} = \frac{V_{,XX}/V}{(V_{,X}/V)^2} = \frac{1}{z^2} \frac{V_{,XX}}{V},$$

the left-hand equation in (26) can also be rewritten as

$$f(z) = \frac{V_{,XX}}{V} - z^2. \quad (27)$$

In the event that Γ cannot be explicitly written as a function of z , then an additional ODE, $\Gamma' = \dots$, is to be considered. However, this case is far more complex and does not frequently arise.

Before going further, we want to point out that the function Γ and the variable z were first identified in [30]. In addition, the dynamical system (22), (23) and (25) was explored in [14] several years before it was studied in [23]. However, it was in [23] where the possibility that for several specific self-interaction potentials, Γ can be explicitly written as a function of z , was explored for the first time. In [14], although the correct dynamical system was identified, the authors were not interested in specific self-interaction potentials. The cost of the achieved generality of the analysis was that the authors had to rely on the obscure notion of ‘instantaneous critical points’.

5. An example: the cosh-like potential

To provide a concrete example of how the function $f(z)$ can be obtained for an specific potential, let us choose the cosh potential [31, 32]

$$V = V_0 \cosh(\mu X). \quad (28)$$

This potential, as well as a small variation of it in the last row of table 1, has a very interesting behaviour near the minimum at $X = 0$. Actually, in the neighbourhood of the minimum, potential (28) approaches

$$V(X) \approx V_0 + \frac{m^2}{2}X^2, \quad m^2 = V_0\mu^2.$$

The quadratic term is responsible for oscillations of the scalar field around the minimum, which play the role of CDM; hence, at late times when the system is going to stabilize in the minimum of $V(X)$, the cosh potential makes the scalar field model of DE indistinguishable from the standard Λ CDM model (see the properties of the equivalent model of [15]).

Taking derivatives of (28) with respect to the scalar field X , we have that $V_{,X} = \mu V_0 \sinh(\mu X)$, and $V_{,XX} = \mu^2 V$. Hence,

$$z = V_{,X}/V = \mu \tanh(\mu X), \quad f(z) = \mu^2 - z^2. \quad (29)$$

Notice that the z , which solves $f(z) = 0$ (i.e., $z = \pm\mu$, which corresponds to the exponential potentials), are the critical points of (25). In table 1, the functions $f(z)$ for several well-known potentials are displayed.

Table 1. The functions, $f(z)$, in equation (27) for different self-interaction potentials.

Correspondence $f(z) \Leftrightarrow V(X)$			
Function $f(z)$	z	Self-interaction potential $V(X)$	Reference
$f(z) = 0$	$\pm\mu$	$V = V_0 e^{\pm\mu X}$	[11]
$f(z) = -\alpha\beta + (\alpha + \beta)z - z^2$	$\frac{\alpha + \beta e^{(\beta-\alpha)X}}{1 + e^{(\beta-\alpha)X}}$	$V = V_0 (e^{\alpha X} + e^{\beta X})$	[21]
$f(z) = (p^2\mu^2 - z^2)/p$	$p\mu \operatorname{cotanh}(\mu X)$	$V = V_0 \sinh^p(\mu X)$	[31]
$f(z) = (p^2\mu^2 - z^2)/p$	$p\mu \tanh(\mu X)$	$V = V_0 \cosh^p(\mu X)$	[31]
$f(z) = (p^2\mu^2 - z^2)/2p$	$p\mu \frac{\sinh(\mu X)}{\cosh(\mu X) - 1}$	$V = V_0 [\cosh(\mu X) - 1]^p$	[32]

Let us collect all of the already-found autonomous ODEs that correspond to a scalar field model of DE with arbitrary self-interaction potential:

$$\begin{aligned}
 x' &= -3x(1 - x^2) + \frac{3\gamma_m}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}y^2z, \\
 y' &= y\left[\frac{3\gamma_m}{2}(1 - x^2 - y^2) + 3x^2\right] + \sqrt{\frac{3}{2}}xyz, \\
 z' &= \sqrt{6}xf(z),
 \end{aligned}
 \tag{30}$$

where it is evident that the equations for x and y are independent of the self-interaction potential, and that details of the given model are encoded in the function $f(z)$ (equation (27)), which depends on the concrete form of the potential (see table 1).

5.1. Critical points

If we apply the simplest tools of dynamical systems theory to the the above example with the cosh potential, the corresponding asymptotic dynamics in the phase space is revealed. Given that, at the minimum of the cosh potential, the model behaves as Λ CDM, here we shall assume that the matter fluid is composed mainly of baryons so that it behaves like pressureless dust ($\gamma_m = 1$).

The following closed system of autonomous ODEs is obtained (it is simply equation (30) with the appropriate substitutions):

$$\begin{aligned}
 x' &= -\frac{3}{2}x(1 - x^2 + y^2) - \sqrt{\frac{3}{2}}y^2z, \\
 y' &= \frac{3}{2}y(1 + x^2 - y^2) + \sqrt{\frac{3}{2}}xyz, \\
 z' &= \sqrt{6}x(\mu^2 - z^2).
 \end{aligned}
 \tag{31}$$

Since $z = \mu \tanh(\mu X) \Rightarrow -\mu \leq z \leq \mu$, the phase space where we look for critical points of (31) is the bounded semi-cylinder (see equation (19)):

$$\begin{aligned}
 \Psi_{\cosh} := \{ &(x, y, z): 0 \leq x^2 + y^2 \leq 1, \\
 &|x| \leq 1, y \geq 0, |z| < \mu\}.
 \end{aligned}
 \tag{32}$$

Nine critical points, $P_i: (x_i, y_i, z_i)$, of the autonomous system of the ODE (31) and one critical manifold are found in Ψ_{\cosh} . These, together with their main properties, are shown in

Table 2. Critical points of the autonomous system of ODE (31) and their properties.

Critical points	x	y	z	Existence	Ω_x	Ω_m	λ_1	λ_2	λ_3
Z	0	0	z	Always	0	1	3/2	-3/2	0
P_{dS}	0	1	0	"	1	0	$-\frac{3}{2} + \frac{3}{2}\sqrt{1 - \frac{4\mu^2}{3}}$	$-\frac{3}{2} - \frac{3}{2}\sqrt{1 - \frac{4\mu^2}{3}}$	-3
P_K^\pm	± 1	0	μ	"	1	0	$3 \pm \sqrt{\frac{3}{2}}\mu$	$\mp 2\sqrt{6}\mu$	3
\bar{P}_K^\pm	± 1	0	$-\mu$	"	1	0	$3 \mp \sqrt{\frac{3}{2}}\mu$	$\pm 2\sqrt{6}\mu$	3
$P_{\bar{X}/m}^\pm$	$\pm \sqrt{\frac{3}{2}}\frac{1}{\mu}$	$\sqrt{\frac{3}{2}}\frac{1}{\mu}$	$\mp \mu$	$\mu^2 \geq \frac{3}{2}$	$\frac{3}{\mu^2}$	$1 - \frac{3}{\mu^2}$	$-\frac{3}{4} + \frac{3}{4}\sqrt{\frac{24}{\mu^2} - 7}$	$-\frac{3}{4} - \frac{3}{4}\sqrt{\frac{24}{\mu^2} - 7}$	6
P_X^\pm	$\pm \frac{\mu}{\sqrt{6}}$	$\sqrt{1 - \frac{\mu^2}{6}}$	$\mp \mu$	$\mu^2 \leq 6$	1	0	$-3 + \mu^2$	$-3 + \frac{\mu^2}{2}$	$2\mu^2$

table 2. For conciseness, here we will concentrate only on those that have some particular interest.

The de Sitter critical point, P_{dS} : $(0, 1, 0)$, for which $\Omega_X = 1$ ($\Omega_m = 0$), is associated with accelerating expansion, since the deceleration parameter

$$q = -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} + \frac{3}{2}(x^2 - y^2) = -1, \quad (33)$$

is a negative quantity. Since $z = 0 \Rightarrow V = V_0$, and since $x = 0 \Rightarrow \dot{X} = 0$, and

$$y = 1 \Rightarrow H = \sqrt{\frac{V_0}{3}} \Rightarrow a(t) \propto e^{Ht},$$

this point corresponds to a de Sitter (inflationary) phase of the cosmic expansion. We underline the fact that to a given point in the phase space, in this case $P_{dS} \in \mathcal{Y}_{\text{cosh}}$, it corresponds to specific cosmological dynamics (de Sitter expansion in the present case).

Next, we have the scalar field X -dominated equilibrium points

$$P_X^\pm: \left(\pm \frac{\mu}{\sqrt{6}}, \sqrt{1 - \frac{\mu^2}{6}}, \mp \mu \right), \quad \Omega_X = 1, \quad q = -1 + \frac{\mu^2}{2}.$$

The first thing one has to pay attention to is the existence of the critical point. By existence, we mean that the given point actually belongs in the phase space ($\mathcal{Y}_{\text{cosh}}$ in the present case). Hence, $|x| \leq 1 \Rightarrow \mu^2 \leq 6$. This is precisely the condition for the existence of the points P_X^\pm : $\mu^2 \leq 6$. In addition, this warrants that $y = \sqrt{1 - \mu^2/6}$ be a real number. If one substitutes (17) back into (15), one gets

$$\omega_X = \frac{x^2 - y^2}{x^2 + y^2},$$

so that for the present case, the EOS parameter of the X -field is $\omega_X = \mu^2/3 - 1 \Rightarrow \gamma_X = \mu^2/3$. Then the continuity equation (1) can be readily integrated at P_X^\pm to obtain $\rho_X = M a^{-3\gamma_X} = M a^{-\mu^2}$, where M is an integration constant. After this, the Friedmann constraint

$$\Omega_X = 1 \Rightarrow 3H^2 = \rho_X \Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{M}{3}} a^{-\mu^2},$$

can also be integrated to obtain the following cosmological dynamics $a(t) \propto t^{2/\mu^2}$, which is to be associated with the critical points, P_X^\pm . Note that whenever $2 < \mu^2 \leq 6$ ($q > 0$), the present scalar-field-dominated solution represents decelerated expansion.

In the case of the matter-dominated critical manifold¹⁸

$$\mathcal{M} = \{(0, 0, z): z \in \mathbb{R}\}, \quad \Omega_m = 1, \quad q = 1/2,$$

instead of an isolated equilibrium point, this is actually a one-dimensional manifold, which is extended along the z -direction. This means, in turn, that the matter-dominated solution, $\Omega_m = 1 \Rightarrow$

$$3H^2 = \rho_m = M a^{-3} \Rightarrow a(t) \propto t^{2/3},$$

may coexist with the scalar field domination no matter whether $V \propto \cosh(\mu X) \rightarrow$ intermediate X , $V = V_0 \rightarrow X = 0$, or $V \propto e^{\pm\mu X} \rightarrow$ very large X ($|X| \rightarrow \infty$). This means that in the phase

¹⁸ Here, by critical manifold we understand a curve in the phase space, all of whose points are critical points.

portrait, a heteroclinic orbit (an orbit connecting two or more different equilibrium points) can be found that joins the matter-dominated equilibrium state with a scalar field domination critical point, originated either by a cosmological constant or by exponential or cosh-like potentials.

5.2. Stability

To judge the stability of given hyperbolic equilibrium points (as explained in the appendix) one needs to linearize the system of ODEs (in this case (31)) which means, in the end, that we have to find the eigenvalues of the linearization or Jacobian matrix, J (see equation (A.2) in the appendix)

$$J = \begin{pmatrix} \partial x'/\partial x & \partial x'/\partial y & \partial x'/\partial z \\ \partial y'/\partial x & \partial y'/\partial y & \partial y'/\partial z \\ \partial z'/\partial x & \partial z'/\partial y & \partial z'/\partial z \end{pmatrix}.$$

Next, J should be evaluated at each non-hyperbolic critical point, such as $J(P_{dS})$, $J(P_X^\pm)$, etc. The eigenvalues of the resulting numerical matrices are what we use to judge the stability of the given points.

Take, for instance, the de Sitter point, P_{dS} : (0, 1, 0). In this case, one must find the roots of the following algebraic equation for the unknown λ :

$$\det \begin{pmatrix} -3 - \lambda & 0 & \sqrt{3/2}\mu \\ 0 & -3 - \lambda & 0 \\ -\sqrt{6}\mu & 0 & -\lambda \end{pmatrix} = 0.$$

The resulting eigenvalues are

$$\lambda_{1,2} = -\frac{3}{2} \pm \frac{3}{2} \sqrt{1 - \frac{4}{3}\mu^2}, \quad \lambda_3 = -3.$$

For $\mu^2 \leq 3/4$, the critical point is an isolated focus, while for $\mu^2 > 3/4$, since the eigenvalues λ_1 and λ_2 are complex numbers with negative real parts, the point P_{dS} is a stable spiral. Hence, the de Sitter solution is always a future attractor (see the bottom panel of figure 1). This is consistent with the fact that the cosh potential is a minimum at $X = 0 \Rightarrow V = V_0$.

The eigenvalues of the linearization matrix, $J(P_X^\pm)$, are

$$\lambda_1 = -3 + \mu^2/2, \quad \lambda_2 = -3 + \mu^2, \quad \lambda_3 = \mu^2.$$

Hence, as required by the existence of $\mu^2 < 6$, P_X^\pm are always saddle critical points (at least one of the eigenvalues is of a different sign).

If one computes the eigenvalues of $J(\mathcal{M})$, one finds $\lambda_{1,2} = \pm 3/2$, $\lambda_3 = 0$. The vanishing eigenvalue is due to the fact that the one-dimensional manifold extends along the z -direction¹⁹. The differing signs of the eigenvalues λ_1 and λ_2 mean that each point in \mathcal{M} is a saddle. It might be a nice exercise for interested readers to judge the stability of the remaining critical points of (31) (see table 2).

¹⁹ In general, a critical point for which at least one of the real parts of the eigenvalues of the Jacobian matrix is vanishing is called a non-hyperbolic critical point. In this case, the Hartman–Grobman theorem [33] cannot be applied. As a consequence, one must go beyond the linear stability theory. The centre manifold theory is useful in this case. For the application of the centre manifold theory in a case of cosmological interest, see [18, 19].

6. Conclusion

In this paper, we have shown in as pedagogical a way as possible how the application of the basic tools of dynamical systems theory can help us understand the dynamics of cosmological models of DE. Our main aim has been to clarify how much useful information about asymptotic cosmological dynamics, which decides the origin and the fate of our Universe, can be extracted by means of these simple tools. We have concentrated our exposition on scalar field cosmological models of DE, and the analysis was kept as general as possible. Special attention has been paid to the derivation of relevant equations and formulas to facilitate the path for beginners to get to concrete computations.

It has been demonstrated that certain generic asymptotic behaviour arises that is quite independent of the concrete model of DE: matter dominance transient phase and dark energy dominance at late times. Just for illustration, we have chosen a concrete potential (the cosh potential), which at late times approaches the Λ CDM model. The approach undertaken here can be easily applied to other self-interaction potentials (see table 1).

In the present paper, for completeness, we have included a very concise and simple exposition of the fundamentals of the theory of dynamical systems, found in the appendix, which may be considered as material for an introductory course on this subject.

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Appendix. Dynamical systems in cosmology

In classical mechanics, the state of a given physical system composed of, say, N particles, is completely specified by the knowledge of the N generalized coordinates, q_i ($i = 1, \dots, N$), and the N conjugated momenta, p_i , which satisfy Hamilton's canonical equations of motion

$$\dot{q}_i = \partial\mathcal{H}/\partial p_i, \quad \dot{p}_i = -\partial\mathcal{H}/\partial q_i,$$

where $\mathcal{H} = \mathcal{H}(q_1, \dots, q_N, p_1, \dots, p_N, t)$ is the Hamiltonian of the physical system. Hence, a given state of this system corresponds to a point in the $2N$ -dimensional space of states spanned by the $2N$ coordinates, $q_1, \dots, q_N, p_1, \dots, p_N$, also known as the 'phase space'.

As for any other physical system, the possible states of a cosmological system may also be correlated with the points in an equivalent state space. However, unlike in the classical mechanics case, the choice of the variables—generalized coordinates and their conjugate momenta—for a cosmological model is not a trivial issue. In this case, a certain degree of uncertainty in the choice of an appropriate set of variables of the phase space arises. There are, however, certain unwritten rules one follows when choosing appropriate variables of the phase space: (i) the variables should be dimensionless, and (ii) whenever possible, they should be bounded. The latter requirement is necessary to have a bounded phase space where all of the existing equilibrium points are 'visible' (i.e., none of them goes to infinity).

The interplay between a cosmological model and the corresponding phase space is possible because of an existing one-to-one correspondence between exact solutions of the cosmological field equation (1) and points in the phase space spanned by given variables, x , y ²⁰:

$$x = x(H, \rho_x, \rho_X), \quad y = y(H, \rho_x, \rho_X).$$

When we replace the original field variables H , ρ_x , and ρ_X by the phase space variables x , y , we have to keep in mind that, at the same time, we trade the original set of nonlinear second-order differential equations with respect to the cosmological time, t (equation (1)), for a set of first-order ODEs):

$$\tilde{x}' = f(x, y), \quad \tilde{y}' = g(x, y), \quad (\text{A.1})$$

where the tilde denotes a derivative with respect to the dimensionless ‘time’ parameter, $\tau \equiv \int da/a \Rightarrow d\tau = d \ln a = H dt$. The introduction of τ instead of the cosmic time t is dictated by simplicity and compactness of writing. In addition, this warrants that the first-order differential equation (A.1) are a closed system of equations.

The most important feature of the system of ODEs (A.1) is that the functions $f(x, y)$ and $g(x, y)$ do not depend explicitly on the parameter τ . This is why (A.1) is called an autonomous system of ODEs. The images of the integral curves of (A.1) in the phase space are called ‘orbits’ of the system of autonomous ODEs. The critical points of (A.1), which are also fixed or stationary points, $P_{cr}: (x_{cr}, y_{cr})$, are those for which the ‘velocity’ vector, $\mathbf{v} = (x', y')$, vanishes (see below):

$$\mathbf{v}(P_{cr}) = (x'(P_{cr}), y'(P_{cr})) = 0.$$

To judge the stability of given equilibrium points, one needs to linearly expand (A.1) in the neighbourhood of each hyperbolic critical point:

$$f(x_{cr} + \delta x, y_{cr} + \delta y), \quad g(x_{cr} + \delta x, y_{cr} + \delta y).$$

Depending on whether the linear perturbations $\delta x = \delta x(\tau)$, $\delta y = \delta y(\tau)$ decay (grow) with time τ or decay in one direction while they grow in the other, the critical point can be a future (past) attractor or a saddle point.

Knowledge of the critical, equilibrium, or fixed points in the phase space corresponding to a given cosmological model is very important. In particular, the existence of attractors can be correlated with generic cosmological solutions that might really decide the fate and/or the origin of the cosmic evolution. The ‘meta-stable’ saddle equilibrium points—stationary points that are not local extrema—can be associated with (no less important) transient cosmological solutions. Hence, in the end we trade the study of the cosmological dynamics depicted by $H = H(t)$, $\rho_x = \rho_x(t)$, $\rho_X = \rho_X(t)$, for the study of the properties of the equilibrium points of the equivalent autonomous system of ODE (A.1) in the phase space, Ψ , of the cosmological model, which, we recall, differs from the usual definition in classical mechanics.

A.1. Remarks on phase space analysis

In this subsection, we provide a very simplified exposition of the fundamentals of dynamical systems theory, which are useful in most cosmological applications [5, 11, 12]. Only knowledge of elementary linear algebra and of ODEs is required to understand this material.

²⁰ Here, for simplicity of the exposition, we consider a two-dimensional phase space.

For a more formal and complete introduction to this subject, we recommend the well-known texts [8, 10, 34–37].

The critical points of the system of ODE (A.1) $P_i: (x_i; y_i)$ (i.e., the roots of the system of algebraic equations)

$$f(x, y) = 0, \quad g(x, y) = 0,$$

correspond to privileged or generic solutions of the original system of cosmological equations. To judge their stability properties, it is first necessary to linearize (A.1) around the hyperbolic equilibrium points²¹. To linearize around a given hyperbolic equilibrium point, P_i , amounts to considering small linear perturbations

$$x \rightarrow x_i + \delta x(\tau), \quad y \rightarrow y_i + \delta y(\tau).$$

These perturbations would obey the following system of coupled ODE (here we use matrix notation):

$$\delta \mathbf{x}' = J(P_i) \cdot \delta \mathbf{x}, \quad \delta \mathbf{x} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}, \quad (\text{A.2})$$

where J , the Jacobian (also linearization) matrix, is to be evaluated at P_i . Thanks to the Hartman–Grobman theorem [33], which basically states that the behaviour of a dynamical system in the neighbourhood of each hyperbolic equilibrium point is qualitatively the same as the behaviour of its linearization, we can safely replace the study of the dynamics of (A.1) with the corresponding study of its linearization (A.2).

We assume that J can be diagonalized (i.e., $J_D = M^{-1}JM$), where M is the diagonalization matrix and

$$J_D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

is the diagonal matrix whose non-vanishing components are the eigenvalues of the Jacobian matrix, J :

$$\det|J - \lambda U| = 0, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{A.3})$$

After diagonalization, the coupled system of ODEs (A.2) is decoupled:

$$\delta \bar{\mathbf{x}}' = J_D \cdot \delta \bar{\mathbf{x}}, \quad \delta \bar{\mathbf{x}} = M^{-1} \delta \mathbf{x}, \quad (\text{A.4})$$

where we have to recall that a different matrix, J_D , corresponds to each equilibrium point. The decoupled system of ODEs (A.4) is easily integrated:

$$\delta \bar{x}(\tau) = \delta \bar{x}(0)e^{\lambda_1 \tau}, \quad \delta \bar{y}(\tau) = \delta \bar{y}(0)e^{\lambda_2 \tau}.$$

Since the diagonal perturbations, $\delta \bar{x}$ and $\delta \bar{y}$ are linear combinations of the perturbations δx , δy :

$$\delta \bar{x} = c_{11} \delta x + c_{12} \delta y, \quad \delta \bar{y} = c_{21} \delta x + c_{22} \delta y,$$

²¹ P_i is a hyperbolic critical point if the real parts of all of the eigenvalues of the linearization matrix around P_i are necessarily non-vanishing. In particular, a hyperbolic point cannot be a centre. In more technical words, a hyperbolic critical point is a fixed point that does not have any centre manifolds.

where the constants c_{ij} are the components of the matrix M^{-1} , then

$$\begin{aligned}\delta x(\tau) &= \bar{c}_{11} e^{\lambda_1 \tau} + \bar{c}_{12} e^{\lambda_2 \tau}, \\ \delta y(\tau) &= \bar{c}_{21} e^{\lambda_1 \tau} + \bar{c}_{22} e^{\lambda_2 \tau},\end{aligned}\tag{A.5}$$

where

$$\begin{aligned}\bar{c}_{11} &= \frac{c_{22} \delta \bar{x}(0)}{c_{22} c_{11} - c_{12} c_{21}}, & \bar{c}_{12} &= -\frac{c_{12} \delta \bar{y}(0)}{c_{22} c_{11} - c_{12} c_{21}}, \\ \bar{c}_{21} &= -\frac{c_{21} \delta \bar{x}(0)}{c_{22} c_{11} - c_{12} c_{21}}, & \bar{c}_{22} &= \frac{c_{11} \delta \bar{y}(0)}{c_{22} c_{11} - c_{12} c_{21}}.\end{aligned}$$

As a matter of fact, we do not need to compute the coefficients \bar{c}_{ij} ; instead, the structure of the eigenvalues, λ_i , is the only thing we need to judge regarding the stability of given (hyperbolic) equilibrium points of (A.1). If the eigenvalues are complex numbers, $\lambda_{\pm} = \nu \pm i\omega$, the perturbations (A.5) do oscillations with frequency ω . If the real part, ν , is positive, the oscillations are enhanced, while if $\nu < 0$, the oscillations are damped. The case $\nu = 0$ is associated with a centre (harmonic oscillations) and is not frequently encountered in cosmological applications. This latter kind of point in the phase space—an eigenvalue with vanishing real part—is called a non-hyperbolic critical point.

A.2. Taxonomy of isolated equilibrium points

Here we summarize the basic classification of isolated equilibrium points in the phase plane. For simplicity, we assume that $\lambda_1 \leq \lambda_2$, but nothing changes if one assumes that $\lambda_1 \geq \lambda_2$, or if one makes no assumption at all. In figure A1, the illustrative phase portraits—the drawing of the trajectories of a dynamical system in the phase plane—are shown.

1. The eigenvalues λ_1, λ_2 in equation (A.5) are real numbers.
 - (a) $0 < \lambda_1 < \lambda_2$. The critical point, P_i , is an unstable node or, also, a source point (past attractor). This is an unstable critical point which represents the origin of a non-empty set of orbits in the phase plane (left-hand top panel of figure A1).
 - (b) $\lambda_1 < 0 < \lambda_2$. P_i is a saddle point. This is an unstable equilibrium point, but it can be associated with a marginally stable state since the orbits of the system of autonomous ODEs (A.1) spend some ‘time’ in the neighbourhood, $N(\epsilon, P_i)$, of the critical point until these leave $N(\epsilon, P_i)$ to go elsewhere in the phase plane (centre top panel of figure A1).
 - (c) $\lambda_1 < \lambda_2 < 0$ (also $\lambda_1 = \lambda_2 < 0$). P_i is a stable node (future attractor). This kind of equilibrium point is associated with an asymptotically stable state. The future attractor is the end point of a non-empty set of phase space orbits generated by a wide range of initial conditions (right-hand top panel of figure A1).
2. The eigenvalues λ_1, λ_2 in equation (A.5) are complex numbers: $\lambda_{1,2} = \nu \pm i\omega$ ($\omega \neq 0$).
 - (a) $\nu > 0$. The equilibrium point, P_i , is an unstable spiral (past attractor). It is associated with unstable oscillation, (i.e., with amplified oscillations, as in the left-hand bottom panel in figure A1).
 - (b) $\nu = 0$. P_i is a centre (not frequent). It is associated with free oscillations (centre bottom panel in figure A1).
 - (c) $\nu < 0$. P_i is a stable spiral (future attractor). The system does damped oscillations until it settles down in the equilibrium state (right-hand bottom panel in figure A1).

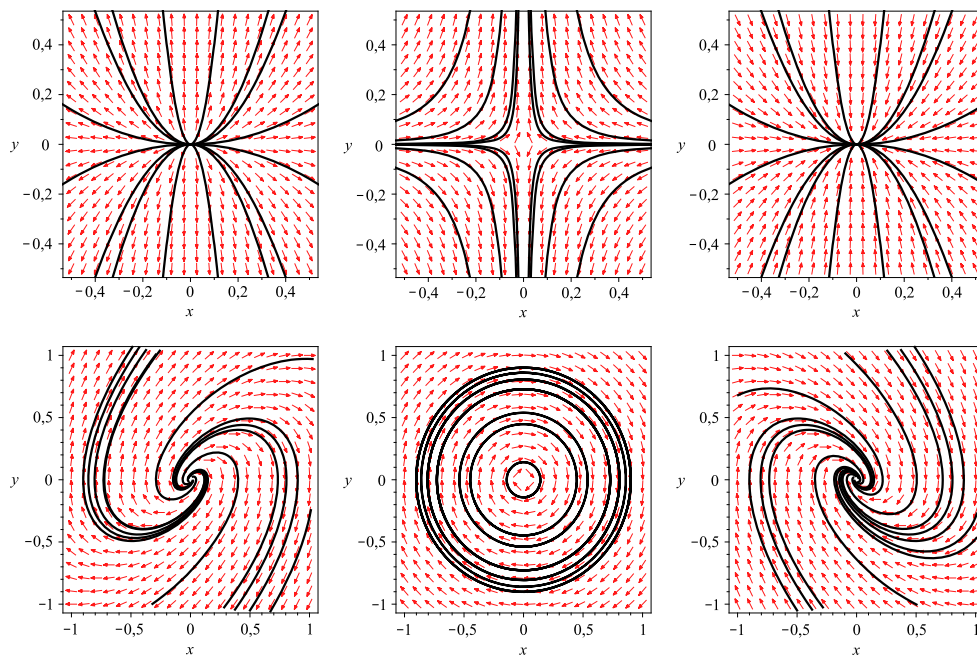


Figure A1. Phase portrait: isolated equilibrium points in the phase space when the eigenvalues λ_1 and λ_2 are real numbers (top panels). The velocity field, represented by the arrow fields in the figures, shows where the flux of the ODE (A.1) flows. Several orbits generated by different sets of initial conditions $[x(0) = x_{0i}; y(0) = y_{0i}]$ are shown in each case. In the left-hand top panel, the origin $P: (0; 0)$ is an unstable node or past attractor ($\lambda_2 > \lambda_1 > 0$), while in the centre top panel the origin is a saddle critical point ($\lambda_1 < 0 < \lambda_2$), and in the right-hand top panel, $P: (0; 0)$ is a stable node or future attractor ($\lambda_1 < \lambda_2 < 0$). In the bottom panels the equilibrium points for cases when the eigenvalues are complex numbers $[\lambda_{1,2} = \nu \pm i\omega]$ are shown. In the left-hand bottom panel, the origin $(x; y) = (0; 0)$ is an unstable spiral ($\nu > 0$), while in the centre panel, it is a centre ($\nu = 0$). The point $(0; 0)$ is a stable spiral in the right-hand bottom panel ($\nu < 0$).

The above classification of isolated critical points encompasses the kinds of equilibrium points most frequently encountered in cosmological applications. Any curve in the phase space that connects critical points (two or more, perhaps all of them) is called a ‘heteroclinic orbit.’ A curve that joins a critical point with itself is called a ‘homoclinic trajectory.’ There are other useful concepts that are not explained here. The interested reader is referred to the bibliography [8, 10, 34–37].

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